IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Rotating fields and particle-like states of electron-positron systems

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1981 J. Phys. A: Math. Gen. 14 597

(http://iopscience.iop.org/0305-4470/14/3/010)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 30/05/2010 at 16:44

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **14** (1981) 597–619. Printed in Great Britain

Rotating fields and particle-like states of electron–positron systems

P F Browne

Department of Pure and Applied Physics, University of Manchester Institute of Science and Technology, Manchester M60 1QD, UK

Received 14 July 1980, in final form 29 October 1980

Abstract. It is argued that the field of a charge in a stationary orbit co-rotates with the charge about the centre of mass. Such a field has no radiation term. It is obtained by transformation from the co-rotating frame of reference; direct calculation would require that the constitutional properties of the medium be known, these being such as to curve light rays. For rotating reference system Corum's tetrad field is employed, the object of anholonomity affecting the derivation of fields from potentials. The force between the charge $\pm e$ of positronium is found to be $(1-\beta^2)^{1/2}\, e^2/r_{12}^2$ when the charges follow with velocity $\boldsymbol{\omega}\times\boldsymbol{r}$ $(=\boldsymbol{\beta}c)$ a stationary circular orbit of diameter r_{12} $(=2r)$.

Stationary orbits are selected by Bohr quantisation of *canonical* angular momentum. A quadratic equation is obtained for $\gamma\beta$ and for r_{12}, where $\gamma=(1-\beta^2)^{-1/2}$. One solution gives the well-known 'atom-like' states. The other gives 'particle-like' states for which orbital motion is ultrarelativistic ($\gamma\beta=n/\alpha$, where $\alpha=1/137$), orbital diameters are small compared with e^2/mc^2 $(\equiv d)$, m being the electron mass, and energies are $2\alpha mc^2/n$, small compared with mc^2. In such a state positronium is termed a 'positronium unit'. The unit has a very long lifetime for radiative transitions, and should possess a weak magnetic moment ($\mu''=\alpha ed/2n$). If it is assigned half-integral orbital angular momentum (hence imaginary parity) it might be identified with the neutrino.

A system in which two units orbit around a stationary charge has the properties of the muon, and a system with one orbiting unit has the properties of the charged pion. Spin, charge, rest energy and decay mode are explained in each case. The very different reactivities also can be explained. A system, 'trionium', in which two charges of one sign orbit about a central stationary charge of the opposite sign is explored. Again ultrarelativistic states emerge, but the energies differ inappreciably from the rest energy mc^2 of the central charge. However, if one adds further pairs of charges in orbits of increasing radius, the magnetic coupling between orbital currents provides energies of order mc^2/α. It is speculated that such a system may afford a model for the proton.

1. Introduction and general philosophy

In the early years of this century physicists were confronted with an empirical classification of atoms based on their chemical reactivity — the periodic table of the elements. In addition, there was some spectroscopic evidence to be explained, chiefly the spectrum of the hydrogen atom and Moseley's law for x-ray lines. The key for the unravelling of this situation was provided by Rutherford when he proposed his electrodynamical model of the atom, and further crucial steps were taken by Bohr when he introduced the concept of stable orbits and thereby found at least a first-order explanation for the hydrogen atom spectrum. One wonders how long progress might have been delayed had not these steps been taken. Today one has an empirical

0305-4470/81/030597+23$01.50 © 1981 The Institute of Physics

classification of particles, again based on reactivity, and something is known about the selection rules governing transitions between particle states. But the mass spectrum of the particles, and hence the reason for their existence, remains a mystery. The reason for the slow progress, I suggest, is the absence of dynamical models of the type that proved so fruitful for atomic physics. Heisenberg (1976) seems to have been moving toward the same conclusion, and after questioning the general philosophy of particle physicists advocated a return to dynamical considerations.

Now of course the word 'fundamental' or 'elementary' is no longer strictly applicable to particles in general. It is recognised that at least the majority are composite systems. In seeking dynamical models for these systems one requires elementary particles; only electrons and positrons, I believe, can fulfil this role. By implication only electromagnetic forces will be involved. The task, then, is to consider how electrons and positrons interact at close range. For two point charges e separated by distance r_{12}, the electrostatic potential energy e^2/r_{12} exceeds the rest energy mc^2 of either charge when $r_{12} \ll d$, where $d = e^2/mc^2 = 2.82 \times 10^{-15}$ m. Electromagnetic interactions in these circumstances may be said to be 'strong'. Motion inevitably is ultrarelativistic, so that a theory capable of dealing with strong interactions is a theory valid in the ultrarelativistic extreme.

It should be emphasised that quantum electrodynamics has not as yet been applied with any generality to strong interactions. The successes of the theory are entirely in the realm of weak interactions, for which the methods of perturbation theory become available. The conceptual problems that arise for strong interactions lie in the mixing of the positive and negative energy states. For a free electron there exists a representation for which the positive and negative energy states are separately described by two-component wavefunctions, and in the case of a weakly interacting electron a representation giving the two-component wavefunctions can still be made; but for strong interactions it fails completely (Foldy and Wouthuysen 1950). Recently the autoionisation of positrons in collisions between highly charged nuclei (Muller *et al* 1972a, b) has necessitated some consideration of strong interactions, albeit rather specialised.

I take the attitude that classical field theory, to which the Wilson–Sommerfeld quantisation rules are added as boundary conditions, is the basic conceptual framework. Quantum mechanics and quantum field theory are viewed as formalisms for taking into account effects due to random fluctuations of electromagnetic radiation field pervading the universe at large. This subject, stochastic electrodynamics, has been receiving increasing attention in recent years. It shows every sign of providing a classical meaning for quantum theory.

Stochastic electrodynamics probably has its origin in Welton's (1948) proof that the radiative corrections of quantum electrodynamics can be obtained by superimposing on the motion of an electron in a local force field a further oscillatory motion due to a randomly fluctuating electromagnetic field. The strength and spectrum of this field are obtained by equating the energy density of the field to the quantum mechanical zero-point energy density—that is $\frac{1}{2}h\nu$ with each standing wave mode of which there are $8\pi\nu^2\,d\nu/c^3$ per unit volume in the frequency interval $\nu \to \nu + d\nu$. The mean potential energy of the electron in the local field alters from $V(\boldsymbol{q})$ to $[1 + \frac{1}{6}\langle(\Delta\boldsymbol{q})^2\rangle\nabla^2 + \ldots]V(\boldsymbol{q})$. Interaction with the random field gives infinite energy, but the difference for two states of the electron is finite. Later Power (1966) showed that the radiative corrections also could be obtained as changes in the energy of the random field due to the presence of the electron system; the latter changes the refractive index of the medium, and the boundary conditions convert a change of light velocity into a frequency change.

Marshall (1963, 1965) has probed more deeply by considering what sort of random field is in equilibrium with a distribution in phase space of charged classical oscillators subject to radiative damping. If the distribution is that of the ground state eigenfunction of the harmonic oscillator, the spectrum of radiation is $4\pi h\nu^3\,d\nu/c^3$. For a mixture of excited state eigenfunctions appropriate for temperature T the radiation has the blackbody spectrum superimposed on the zero-point spectrum. For recent reviews of this subject see Boyer (1975) and de la Pena and Cetto (1979). The random field will necessitate a statistical averaging over classical particle trajectories. Concerning this subject see Feynman (1948), Motz (1962) and de la Pena-Auerbach and Garcia-Colin (1968).

The random fluctuations of electromagnetic field pervading the universe are one phenomenon underlying the quantum mechanical formalism. Another is the discreteness of action in any measurement. In the old quantum theory a canonical transformation is used to introduce angle–action variables φ_k and J_k, and since the Hamiltonian is independent of φ_k in the case of multiply periodic motion, the Hamilton equations of motion yield immediately $J_k=$ constants and $\dot{\varphi}_k=$ constants. Writing $J_k=n_k h$, where h is Planck's constant and n_k are integers, the discreteness of action is expressed. But up to this point classical electrodynamics deals with the motion of particles, and now there arises the question as to what constitutes a particle. Accepting that only electrons and positrons are elementary, the answer must be some sort of singularity in the electromagnetic field. Although action-at-a-distance formulations of electrodynamics may be possible, one deals fundamentally with a continuum of field energy. In a continuum discreteness can arise only in the form of standing wave modes, which in turn require boundary conditions. The Wilson–Sommerfeld quantisation rules must be introduced in the role of boundary conditions.

Particularly revealing for the understanding of quantisation is the Bohm–Aharanov effect, whose interpretation still is controversial (Erlichson 1970). Let a beam of electrons of momentum $\boldsymbol{p}$ and energy E pass through a pair of slits at A and B and then strike a screen. What determines whether an electron will be detected at point P on the screen? Permitting a vector potential field $\boldsymbol{A}$ the answer depends on the integral $\oint(\boldsymbol{p}+e\boldsymbol{A}/c)\cdot d\boldsymbol{l}$, where the closed path is PABP. For an electron to be detected this has to be an integral multiple of h; because of the random field this condition does not determine P precisely, but only statistically. It is possible to change $\oint\boldsymbol{A}\cdot d\boldsymbol{l}$ without exerting any Lorentz force on the electrons, and when this is done P is found to vary. Thus properties of the medium must affect where the electron is detected; specifically, the detection process requires a boundary condition on the superimposed fields of the electron and of the medium, and changes to the latter thereby influence where the singularity is detected.

The simplest system of electrons and positrons, positronium, will be considered in this paper in the light of a modified stationary orbit field. The classical electrodynamics of this simple system continues to be a topic for discussion; restricting attention to circular motion (as opposed to one-dimensional motion), much of the literature can be traced from papers by Schild (1963), Schild and Schlosser (1965, 1968), Staruszkiewicz (1968), Synge (1972), Andersen and von Baeyer (1972), Bruhns (1973), Schild (1975, 1976), Fahnline (1977), Kracklauer (1978), Stephas (1978) and Fujigaki and Kojima (1978).

The two-body system highlights a problem which is general to relativistic mechanics. Given a system of particles with world lines $x^i_{(k)}(\lambda_{(k)})$, where k specifies the particle, $\lambda_{(k)}$ is a parameter, and x^i are space–time coordinates, at what times does one consider the positions of the particles when evaluating the momentum and energy of the system? If one locates the particles on the hyperplane of constant coordinate time $x^4_{(o)}$ = constant, where o is a particular particle (at rest relative to the observer), then it is not possible to have interaction between the particles as well as covariance of the world lines (Van Dam and Wigner 1966, Leutwyler 1965, Currie 1966). The reason is that the relative positions are not specified in an invariant manner. The problem can be avoided by abandoning the use of a universal coordinate time in favour of a universal proper time λ. Instead of a single coordinate time and as many proper times as there are particles, one introduces a single proper time and as many coordinate times as there are particles. Treating the latter as independent variables like the position coordinates, interacting pairs of particles may be constrained to have null separation by means of a Lagrangian multiplier term in the action principle (Fahnline 1977). The number of independent variables in a relativistic canonical formalism then becomes $8N$ rather than $6N$, but of course there are constraints such as $v^i_{(k)}v_{i(k)} = 1$ (Dirac 1950, Rohrlich 1979). Also it is to be expected that spin effects are implicit in a relativistic theory.

In this paper the primary concern is why an electron in a stationary orbit fails to radiate. Little attention seems to have been given to this problem, although the question of whether to assume a time-asymmetrical or a time-symmetrical field for charge 2 at the position of charge 1 has been much discussed (e.g. Driver 1963, Hill 1967, Staruszkiewicz 1970). In other words, does charge 1 respond to the retarded field of charge 2 which at the same instant responds to the advanced field of 1 (time-asymmetrical interaction), or does charge 1 respond to the arithmetic mean of the retarded and advanced fields of 2 (time-symmetrical interaction)? Probably the interaction is time-asymmetrical or time-symmetrical depending on whether radiative reaction is included or excluded from the equations of motion (Browne 1969). In the time-symmetrical formulation the radiation field of the charge is cancelled by a field generated by the absorber (Wheeler and Feynman 1945). This may be the reason why the charge in a stationary orbit does not radiate, but there remains the question of what to assume for the resultant field. It will be suggested that the field is that which results from transformation from the co-rotating frame of reference. The field will be found by adapting a method due to Corum (1977, 1980).

It will be concluded that the force between the charges of positronium is $(1-\beta^2)^{1/2}e^2/r_{12}^2$, where charges $\pm e$, each with velocity $\boldsymbol{\omega}\times\boldsymbol{r}$ $(\equiv\boldsymbol{\beta}c)$, follow the same circular orbit of radius r and diameter r_{12} equal to the instantaneous separation of the particles for the centre-of-mass observer. The canonical angular momentum of the system is found to be $2r\gamma\beta mc(1-u)$, where $u = d/r_{12}$ with $d = e^2/mc^2$ and $\gamma = (1-\beta^2)^{-1/2}$. After Bohr quantisation of the latter, a quadratic equation is obtained for either $\gamma\beta$ or u, or equivalently r_{12}. One solution gives the familiar 'atom-like' states of positronium. The second solution gives new states, which might be termed 'particle-like'. For the latter, motion is ultrarelativistic, orbital diameters are smaller than d by the same factor that the orbital diameters of the atom-like states are greater than d, and energies differ from zero (not $2mc^2$) by order $mc^2/137$. This is not the first time a second solution has been obtained (Milne 1948, Sternglass 1961, 1965, Smith 1965, Browne 1966), but the states now obtained differ radically from those found previously.

In an ultrarelativistic state positronium is termed a 'positronium unit'. Properties of the unit are examined in § 6. It is found to be extremely stable against radiative decay into photons, because the wavelength would be so much larger than the orbital diameter. Although uncharged, a small magnetic moment is found for the singlet state.

However, the spin of a unit may not be zero; there is reason to assign spin $\frac{1}{2}$ to a unit, and the possibility of identifying it with a neutrino is considered.

In §§ 7 and 8 two further electron–positron systems are considered with a view to obtaining models for sub-nuclear particles. Preliminary investigations of the systems encourage the view that all matter may have an electron–positron structure.

2. Rotating electromagnetic fields

If a charge has uniform linear motion its field may be evaluated directly from the Lienard–Wiechert potentials, or indirectly by Lorentz-transforming the field measured in the rest frame of the charge. In the case of a charge following a circular orbit one might expect again to have the option of two methods for calculation of the field. If the direct evaluation again uses the Lienard–Wiechert potentials for a medium with rectilinear light propagation, the result will be found to differ from the field obtained by transforming the field measured in the co-rotating frame. There are now two possibilities for the field; which does one accept?

This question underlies the Oppenheimer–Schiff paradox (Schiff 1939, Corum 1977). Specifically, let a concentric sphere capacitor be charged and then rotated. If the rotation is with respect to an inertial frame, there arises outside the capacitor a magnetic field because the magnetic fields due to the rotation of the opposite charges on the two spheres do not quite cancel. On the other hand, if the rotation is induced by adoption of a rotating (i.e. non-inertial) reference system, then the external magnetic field does vanish; this follows because the transformation of coordinates cannot alter observables, and both the electric and the magnetic fields vanish for the inertial system. Schiff formally resolves the paradox in terms of the off-diagonal elements of the metric tensor for the rotating reference system. Corum uses a different type of rotating reference system, a tetrad field formed from the instantaneous rest frames of particles of an imaginary rotating fluid; such a reference system is anholonomic essentially because of the field of time scales in use, and the object of anholonomity plays an essential role in obtaining the result $\boldsymbol{B} = \boldsymbol{\beta} \times \boldsymbol{E}$, which suffices to resolve the paradox (since it implies that B vanishes when E vanishes).

Thus there are two fields to consider when dealing with a charge in a circular orbit. Using the terms 'rotating' and 'stationary' to mean relative to one's reference system, one must distinguish between (i) rotation of the charge with the background matter stationary, and (ii) rotation of the charge with co-rotation of the background matter. The fields of the charge differ for these two situations, as the Oppenheimer–Schiff paradox illustrates. I offer the hypothesis that when the orbit is stationary the field is of type (ii), not type (i) as currently assumed. The field in situation (ii) has no radiation term, whereas the field in situation (i) has such a term.

The plausibility of the hypothesis can best be seen by treating the radiation field of an accelerated charge as a transverse disturbance in the Faraday tubes of flux that emanate from the charge. An argument due to Stokes for explaining the production of x-rays is particularly illuminating. Let a charge e initially have uniform velocity v, and at time t_1 be brought to rest by large deceleration a lasting for time Δt. Let the rest position of the charge be Q. Initially the flux tubes co-move with velocity v. At a time $t_1 + t$ consider the configuration of a tube which makes angle θ with the direction of the original motion (and deceleration). Out to distance r, where $r = ct$, the tube will have taken up the position appropriate for the new steady state, namely the charge at rest at

Q. Beyond distance $r + c\Delta t$ the tube still has velocity v, and is directed toward a point Q′, which is where the charge would have reached had the original motion continued. If the change in motion were instantaneous, the outward propagating disturbance would amount to two discontinuities in the direction of the tube at points A and C in figure 1,

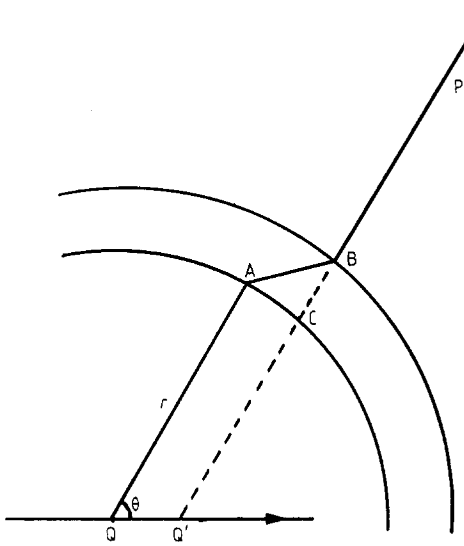

Figure 1. AB represents the outward propagating disturbance to the flux tube QABP. Q is the rest position of the charge, and Q′ the position it would have attained had it not been brought to rest.

the tube wrapping itself around a sphere of radius r for a distance AC. In reality the charge is brought to rest in the finite time Δt, and so the tube adopts the configuration QABP in figure 1. Since $AC = QQ' \sin\theta$, where $QQ' = vt = a\Delta t r/c$ and $BC = c\Delta t$, we have $AB/BC = ar \sin\theta/c^2$. This is the ratio of the transverse to the longitudinal field, and since the latter is e/r^2 the former must be $ea \sin\theta/rc^2$. This is the radiation field caused by the change in motion of the charge. It falls off as $1/r$ rather than $1/r^2$ because the outer tube continues to move with the original velocity v for the delay time r/c, and the farther out the greater is the displacement incurred in this time. If the tubes were to have the configuration which results from transformation from the co-rotating frame of reference, transverse disturbances of the above type will not arise, but of course the tubes will be curved and light rays will be curved (Browne 1977a). Moreover, the retarded position of charge e_2 for the field at the position of e_1 will be affected by curvature of the light rays; the light ray from the retarded position of e_2 to the present position of e_1 passes through the centre of mass 0, which is what one expects instinctively but not what Schild (1963) has envisaged.

It is possible to derive the full relativistically correct field for an accelerated charge without resort to the Lienard–Wiechert solution of Maxwell's equations, and it is particularly instructive in the present context to do so. The argument is given by Page and Adams (1940), who develop the entire theory of electromagnetism from what they term the 'emission theory'. 'Emitters' distributed over the surface of the electron are assumed to deliver streams of 'moving elements' which are disturbances propagating with the velocity of light $\boldsymbol{c}$. Each emitter delivers its own stream of moving elements rather as a machine gun delivers a stream of bullets. The locus of moving elements from a particular emitter, considered simultaneously for frame S, is a field line for S. Due to the relativity of simultaneity, events on a field line for S do not lie on a field line for another frame S′. The electric field $\boldsymbol{E}$ is the number of field lines per unit area normal to the field direction at the point in question. The magnetic field $\boldsymbol{B}$ is defined by $\boldsymbol{B} = \boldsymbol{c} \times \boldsymbol{E}/c$. The entire theory of electromagnetism then follows from the relativistic kinematics of moving elements.

Regarding the field of an accelerated charge, the essential steps in the argument briefly are as follows. Let element 1 leave an emitter at time t_0 when the position of the electron is Q_1, and let it propagate distance r to P_1 reached at time $t_0 + r/c$ $(= t)$. Over the same time interval the electron moves distance $\boldsymbol{v}\ \mathrm{d}t_0$ to Q_2 where element 2 departs from the same emitter and has time to propagate to P_2. Because the direction of the emitter may change, the velocities of the moving elements are $\boldsymbol{c}$ and $\boldsymbol{c} + \mathrm{d}\boldsymbol{c}$, where of course $\boldsymbol{c} \cdot \mathrm{d}\boldsymbol{c} = 0$. We require to calculate P_2P_1 $(= \mathrm{d}\boldsymbol{l})$. Clearly

$$\mathrm{d}\boldsymbol{l} = r\boldsymbol{c}/c - \boldsymbol{v}\ \mathrm{d}t_0 - (\boldsymbol{c} + \mathrm{d}\boldsymbol{c})(r/c - \mathrm{d}t_0) = (\boldsymbol{c} - \boldsymbol{v})\ \mathrm{d}t_0 - r\ \mathrm{d}\boldsymbol{c}/c. \tag{1}$$

The number of field lines crossing unit area normal to $\boldsymbol{c}$ can be found from the relativistic aberration formulae, because the angular distribution of emitters is uniform in the instantaneous rest frame and the total number for a charge of strength e is $4\pi e$. The result is

$$n = \frac{(1 - v^2/c^2)e}{(1 - \boldsymbol{v} \cdot \boldsymbol{c}/c^2)r^2}. \tag{2}$$

We require the number per unit area normal to $\mathrm{d}\boldsymbol{l}$, so that we divide by $\boldsymbol{c} \cdot \mathrm{d}\boldsymbol{l}/c\ \mathrm{d}l$, bearing in mind that the field has the direction of $\mathrm{d}\boldsymbol{l}$. Then

$$E = n\left(\frac{c\ \mathrm{d}l}{\boldsymbol{c} \cdot \mathrm{d}\boldsymbol{l}}\right)\left(\frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}l}\right) = \frac{e(1 - v^2/c^2)}{r^2(1 - \boldsymbol{v} \cdot \boldsymbol{c}/c^2)^3}\left(\boldsymbol{c} - \boldsymbol{v} - \frac{r\ \mathrm{d}\boldsymbol{c}}{c\ \mathrm{d}t_0}\right)c^{-1}. \tag{3}$$

The term involving $\mathrm{d}\boldsymbol{c}/\mathrm{d}t_0$ gives the radiation field. In S two effects contribute to $\mathrm{d}\boldsymbol{c}/\mathrm{d}t_0$: the changing angle of aberration, and the Thomas precession of the emitters. By considering the emission of the successive moving elements with respect to S′, the instantaneous rest frame when the first is emitted, one may eliminate the Thomas precession. With respect to S′ the velocities are $\boldsymbol{c}'$ and $\boldsymbol{c}' + \mathrm{d}\boldsymbol{c}'$. One requires that $\boldsymbol{c}' + \mathrm{d}\boldsymbol{c}'$ should transform into $\boldsymbol{c}''$ $(= \boldsymbol{c}')$ for the instantaneous rest frame S″ when the second is emitted. This gives, to first order in the relative velocity of S″ and S′ divided by c',

$$c'' = \frac{\boldsymbol{c}' + \mathrm{d}\boldsymbol{c}' - \boldsymbol{a}'\ \mathrm{d}t_0'}{1 - \boldsymbol{c}' \cdot \boldsymbol{a}'\ \mathrm{d}t_0'/c'^2} \simeq \boldsymbol{c}' + \mathrm{d}\boldsymbol{c}' - \frac{\boldsymbol{c}' \times (\boldsymbol{a}' \times \boldsymbol{c}')\ \mathrm{d}t_0'}{c'}. \tag{4}$$

Hence

$$\mathrm{d}\boldsymbol{c}'/\mathrm{d}t_0' = \boldsymbol{c}' \times (\boldsymbol{a}' \times \boldsymbol{c}')/c'. \tag{5}$$

It remains to transform this equation to S. This can be done directly, or use can be made of a result due to Page (1929) for the rate of change of a vector with respect to a rotating frame. In the latter case one writes (5) as $\mathrm{d}\boldsymbol{c}'/\mathrm{d}t_0' = \boldsymbol{\omega}' \times \boldsymbol{c}'$, where $\boldsymbol{\omega}' = \boldsymbol{c}' \times \boldsymbol{a}'/c'$, and obtains with respect to S $\mathrm{d}\boldsymbol{c}/\mathrm{d}t_0 = \boldsymbol{\omega} \times \boldsymbol{c}$, where $\boldsymbol{\omega} = (\boldsymbol{c} - \boldsymbol{v}) \times \boldsymbol{a}/(c^2 - v^2)$. When this result is inserted into (3), the field agrees with that evaluated from the Lienard–Wiechert potentials.

One notes the origin of the radiation field — the rate of change of the angle of aberration for a moving element with respect to an emitter, and the Thomas precession of the emitter. One notes also that the result (3) depends on rectilinear light rays.

Knowing the field with respect to a frame S′ which is inertial in the rotational sense, one requires to calculate the field with respect to a reference system R which rotates relative to S′. For R we choose, following Corum (1977, 1980), a field of Lorentz frames which are instantaneously at rest relative to particles of an imaginary rotating fluid — a fluid which has rotational velocity field $-\boldsymbol{\beta}(r)c$ relative to S′. If (r', θ', z') is a cylindrical polar coordinate system in S′, and t' coordinate time in S′, and if tangents to these coordinate curves define at each point an orthonormal tetrad of basis vectors $e'_\alpha = (\delta/\delta r', \delta/r'\delta\theta', \delta/\delta z', \delta/c\delta t')$, then the non-inertial tetrad field R is obtained by rotation in the $\theta' - t'$ plane of the above tetrads through angle $\tanh^{-1}\beta$. That is,

$$e_\alpha = \left(\frac{\delta}{\delta r'}, \frac{\gamma}{r}\frac{\delta}{\delta\theta'} - \frac{\gamma\beta}{c}\frac{\delta}{\delta t'}, \frac{\delta}{\delta z'}, \frac{\gamma}{c}\frac{\delta}{\delta t'} - \frac{\gamma\beta}{r'}\frac{\delta}{\delta\theta'}\right), \tag{6}$$

and the dual 1-forms are

$$\boldsymbol{\omega}^\alpha = (\mathbf{d}r', \gamma r'\,\mathbf{d}\theta' + \gamma\beta c\,\mathbf{d}t', \mathbf{d}z', \gamma c\,\mathbf{d}t' + \gamma\beta r'\,\mathbf{d}\theta'). \tag{7}$$

By use of

$$\mathbf{d}\boldsymbol{\omega}^\alpha = 2\Omega^\alpha_{\beta\gamma}\boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma, \tag{8}$$

one may evaluate the components of the object of anholonomity for the reference system, obtaining for the non-zero components

$$\begin{aligned} 2\Omega^2_{12} &= \gamma^2/r', & 2\Omega^2_{14} &= \gamma^2(\delta\beta/\delta r' - \beta/r'), \\ 2\Omega^4_{12} &= \gamma^2(\delta\beta/\delta r' + \beta/r'), & 2\Omega^4_{14} &= -\gamma^2\beta^2/r'. \end{aligned} \tag{9}$$

Since Ω^k_{ij} is the antisymmetric part of the affine connections, the electromagnetic field tensor F_{ij} is related to the 4-potential A^i by

$$F_{ij} = A_{j,i} - A_{i,j} + 2\Omega^k_{ij}A_k. \tag{10}$$

The 4-potential A^i is obtained by Lorentz transformation of the 4-potential A'^i appropriate for velocity $-\beta c$:

$$A_\theta = \gamma A'_\theta + \gamma\beta\varphi', \qquad \varphi = \gamma\varphi' + \gamma\beta A'_\theta. \tag{11}$$

It remains only to substitute (9) and (11) into (10). With the help of

$$(F_{14}, F_{24}, F_{34}) = -\boldsymbol{E}, \qquad (F_{23}, F_{31}, F_{12}) = -\boldsymbol{B}, \qquad \boldsymbol{E}' = -\boldsymbol{\nabla}'\varphi', \qquad \boldsymbol{B}' = \boldsymbol{\nabla}' \times \boldsymbol{A}', \tag{12}$$

one obtains

$$\boldsymbol{E}_r = \gamma(\boldsymbol{E}' - \boldsymbol{\beta} \times \boldsymbol{B}')_r, \qquad \boldsymbol{B}_z = \gamma(\boldsymbol{B}' + \boldsymbol{\beta} \times \boldsymbol{E}')_z. \tag{13}$$

The result (13) is what would be expected for two frames with constant relative velocity. Now we have obtained it for the 4-potential $(0, A'_\theta, 0, \varphi')$ with β an unspecified function of r'. It is remarkable that the terms involving $\delta\beta/\delta r'$ are cancelled by terms arising from $\Omega^k_{ij}A_k$ in (10).

A charge e which is at rest in S′ has velocity $\boldsymbol{\beta}c$ in R. It is subject to the Lorentz force

$$\boldsymbol{F} = e(\boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{B}) = e(1 - \beta^2)\boldsymbol{E}, \tag{14}$$

since (13) implies that $\boldsymbol{B} = \boldsymbol{\beta} \times \boldsymbol{E}$. This result will be of considerable importance.

Now consider positronium. Charges e_1 and e_2, each with rest mass m, follow with velocity $\boldsymbol{\omega} \times \boldsymbol{r}$ a circular orbit of radius r. It is assumed that the charges occupy diametrically opposed positions at any instant in the rest frame of the centre of mass, so that the separation between the charges is the orbital diameter ($r_{12} = 2r$). In general the charges might follow confocal ellipses, the centre of mass 0 being the common focus, but attention will be restricted to the case when the two ellipses degenerate into a single circle.

S′ is the co-rotating frame with respect to which e_1 and e_2 are at rest. Their mutual attraction e_1e_2/r_{12}^2 is balanced by an outward gravitational force on each particle, the source of the gravitational field being the rotating background matter. With respect to S′ the 4-potential due to e_2 at the position of e_1 has the components (0, 0, 0, e_2/r'_{12}). With respect to R the 4-potential will have components given by (11), namely (0, A_θ, 0, φ), where

$$A_\theta = \gamma_1\beta_1 e_2/r_{12}, \qquad \varphi = \gamma_1 e_2/r_{12} \tag{15}$$

and the fields, obtained by putting $E'_r = -e_2/r'^2_{12}$ and $B' = 0$ in (13), are

$$E_r = \gamma_1 e_2/r_{12}^2, \qquad B_z = -\gamma_1\beta_1 e_2/r_{12}^2. \tag{16}$$

Note that the velocity entering into (15) and (16) is not that of the source charge e_2, but rather the velocity of the field lines at the position of e_1.

The Lorentz force between e_1 and e_2, obtained by substitution from (16) into (14), is

$$F = (1-\beta_1^2)^{1/2} e_1 e_2/r_{12}^2. \tag{17}$$

This will be recognised as the force between two charges with constant velocity in the same direction along parallel straight lines normal to the separation r_{12} of the charges — for example, the force between two charges attached to nylon threads which are being wound onto the same spool and whose separation is a vector normal to the threads. In the case of such linear motion clearly neither charge moves across the electric field lines of the other charge. The same will be true for circular motion when the field lines co-rotate with the charges about the centre of mass 0.

3. Interaction momentum and energy

Let the fields ($\boldsymbol{E}_1$, $\boldsymbol{B}_1$) due to charge e_1 alone be superimposed on the fields ($\boldsymbol{E}_2$, $\boldsymbol{B}_2$) due to charge e_2 alone, yielding resultant fields ($\boldsymbol{E}$, $\boldsymbol{B}$). The energy density in the field is $k(E^2+B^2)$, where $k = (8\pi)^{-1}$. Since $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$ and $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$, the energy density can be split into four terms. $k(E_1^2 + B_1^2)$ contributes to the rest energy of charge e_1, and $k(E_2^2+B_2^2)$ to that of e_2; indeed the rest energies may be wholly electromagnetic (Rohrlich 1960). The interaction energy is $2k(\boldsymbol{E}_1 \cdot \boldsymbol{E}_2 + \boldsymbol{B}_1 \cdot \boldsymbol{B}_2)$. By use of

$$\boldsymbol{E} = -\delta \boldsymbol{A}/c\delta t - \boldsymbol{\nabla}\varphi, \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \tag{18}$$

the electrostatic interaction energy ΔW_E and the magnetic interaction energy ΔW_B may be expressed in terms of the potentials. One finds

$$\begin{aligned} \Delta W_E &= 2k\int \boldsymbol{E}_1 \cdot \boldsymbol{E}_2 \,\mathrm{d}^3x = -2k\int \boldsymbol{\nabla}\varphi_1 \cdot \boldsymbol{E}_2 \,\mathrm{d}^3x = 2k\int \varphi_1 \boldsymbol{\nabla} \cdot \boldsymbol{E}_2 \,\mathrm{d}^3x \\ &= \int \varphi_1\rho_2 \,\mathrm{d}^3x = \int e_2\delta(\boldsymbol{r}-\boldsymbol{r}_2)\varphi_1 \,\mathrm{d}^3x = e_2\rho_1(\boldsymbol{r}_2) \end{aligned} \tag{19}$$

and

$$\Delta W_B = 2k \int \boldsymbol{B}_1 \cdot \boldsymbol{B}_2 \, \mathrm{d}^3x = 2k \int \nabla \times \boldsymbol{A}_1 \cdot \boldsymbol{B}_2 \, \mathrm{d}^3x = 2k \int \boldsymbol{A}_1 \cdot \nabla \times \boldsymbol{B}_2 \, \mathrm{d}^3x$$
$$= \int \boldsymbol{A}_1 \cdot \boldsymbol{j}_2 \, \mathrm{d}^3x = \int \boldsymbol{A}_1 \cdot \boldsymbol{\beta}_2 e_2 \delta(\boldsymbol{r} - \boldsymbol{r}_2) \, \mathrm{d}^3x = e_2 \boldsymbol{\beta}_2 \cdot \boldsymbol{A}_1(\boldsymbol{r}_2). \tag{20}$$

In (19) one assumes that $\varphi_1 \boldsymbol{E}_2$ vanishes on a boundary surface, and in (20) $\boldsymbol{A}_1 \times \boldsymbol{B}_2$ is assumed to vanish on the boundary surface. Thus

$$\Delta W = \Delta W_E + \Delta W_B = e_2(\rho_1 + \boldsymbol{\beta}_2 \cdot \boldsymbol{A}_1). \tag{21}$$

If one uses (15) for the potentials it follows that $W = \gamma_2(1 + \beta_2^2) e_1 e_2 / r_{12}$. But the result is questionable because (18) is inconsistent with (10).

The momentum density in the field may be treated similarly. The interaction contribution is $(2k/c)(\boldsymbol{E}_2 \times \boldsymbol{B}_1 + \boldsymbol{E}_1 \times \boldsymbol{B}_2)$. Again with the help of (18) one finds

$$\Delta \boldsymbol{p}_1 = (2k/c) \int \boldsymbol{E}_2 \times \boldsymbol{B}_1 \, \mathrm{d}^3x = -(2k/c) \int \nabla \varphi_2 \times \boldsymbol{B}_1 \, \mathrm{d}^3x = (2k/c) \int \varphi_2 \nabla \times \boldsymbol{B}_1 \, \mathrm{d}^3x$$
$$= c^{-1} \int \varphi_2 \boldsymbol{j}_1 \, \mathrm{d}^3x = \int \varphi_2 e_1 \boldsymbol{\beta}_1 \delta(\boldsymbol{r} - \boldsymbol{r}_1) \, \mathrm{d}^3x = e_1 \boldsymbol{\beta}_1 \varphi_2(\boldsymbol{r}_1). \tag{22}$$

Similarly, $\Delta \boldsymbol{p}_2 = e_2 \boldsymbol{\beta}_2 \varphi_1(\boldsymbol{r}_2)$. $\Delta \boldsymbol{p}_1$ may be termed the 'potential momentum' of e_1 in the field of e_2, analogous to the potential energy of e_1 in the field of e_2. The sum of the potential momenta of e_1 and e_2 may vanish, but the 'potential angular momenta' add.

Following the same steps, the interaction angular momentum is

$$(2k/c) \int \boldsymbol{r} \times (\boldsymbol{E}_2 \times \boldsymbol{B}_1 + \boldsymbol{E}_1 \times \boldsymbol{B}_2) \, \mathrm{d}^3x = \boldsymbol{r}_1 \times \Delta \boldsymbol{p}_1 + \boldsymbol{r}_2 \times \Delta \boldsymbol{p}_2 + I, \tag{23}$$

where

$$I = \int (\boldsymbol{r} \cdot \nabla)(\varphi_1 \boldsymbol{B}_2 + \varphi_2 \boldsymbol{B}_1) \, \mathrm{d}^3x. \tag{24}$$

One notes that so far as the interaction angular momentum is concerned, the potential momentum can be considered to be located at the source of the $\boldsymbol{B}$ field, and the sum of the moments of the potential momenta of the charges equals the interaction angular momentum plus I. If $\varphi_1 \boldsymbol{B}_2 + \varphi_2 \boldsymbol{B}_1$ is an odd function of $\boldsymbol{r}$, the term I will vanish.

There exists a substantial literature on the question of interaction energy, momentum, and angular momentum. For action-at-a-distance formulation of relativistic mechanics the interaction contributions are required in order to avoid 'no interaction' theorems; for expressions in this approach see Dettman and Schild (1954), Schild (1963, 1975, 1976), Van Dam and Wigner (1965, 1966) and Currie (1966). A more direct and satisfying approach, that followed above, is to integrate the densities of the interaction quantities in the electromagnetic field, and this approach also has received widespread attention (Furry, 1969, Calkin 1966, 1971, Konopinski 1978, Stephas 1978, Graham and Lahoz 1980).

4. Lagrangian

The basic equation of relativistic classical electrodynamics is

$$\mathrm{d}(\gamma\boldsymbol{\beta}mc)/\mathrm{d}t = e(\boldsymbol{E}+\boldsymbol{\beta}\times\boldsymbol{B}), \tag{25}$$

which describes the motion of a charge e with mass m and velocity $\boldsymbol{\beta}c$ in an electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$. Radiative reaction has been omitted. In the case of circular orbits of positronium one substitutes the rotating fields (16) when the orbits are stationary. The interparticle force then is (17), and the equation of motion (25) is

$$\gamma\beta^2mc^2/r = \gamma(1-\beta^2)e^2/r_{12}^2. \tag{26}$$

On defining

$$u = d/r_{12}, \qquad d = e^2/mc^2 = 2.82\times10^{-13}\ \mathrm{cm} \tag{27}$$

and noting that $r_{12}=2r$, one may reduce (26) to

$$2\gamma^2\beta^2 = u. \tag{28}$$

If one assumes that the fields and the potentials are related by (18), not (10), one may rewrite (25) in the form,

$$\mathrm{d}\boldsymbol{p}/\mathrm{d}t = -e\boldsymbol{\nabla}(\varphi-\boldsymbol{\beta}\cdot\boldsymbol{A}), \qquad \boldsymbol{p} = \gamma\boldsymbol{\beta}mc + e\boldsymbol{A}/c. \tag{29}$$

To see this one notes that $\mathrm{d}\boldsymbol{A}/c\ \mathrm{d}t = \delta\boldsymbol{A}/c\delta t + (\boldsymbol{\beta}\cdot\boldsymbol{\nabla})\boldsymbol{A}$ and then uses $\boldsymbol{\beta}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\beta}\cdot\boldsymbol{A})-(\boldsymbol{\beta}\cdot\boldsymbol{\nabla})\boldsymbol{A}$. The inertial term in (29) incorporates interaction momentum $e\boldsymbol{A}/c$ as well as kinetic momentum $\gamma\boldsymbol{\beta}mc$ (see equation (19)). Then the equality of action and reaction is assured, and hence the independence of internal and centre-of-mass motions (Breitenberger 1968). In the form (29) the equations of motion are the Euler–Lagrange equations for the variational principle,

$$\delta\int L\ \mathrm{d}t = 0, \qquad L = -mc^2(1-\beta^2)^{1/2} + e(\varphi-\boldsymbol{\beta}\cdot\boldsymbol{A}). \tag{30}$$

The Cartesian coordinate variables (x, y, z) and the momentum variables (p_x, p_y, p_z) are canonically conjugate.

If the potentials in (30) are as given by (15), the Lagrangian reduces to the simple form

$$L = -mc^2(1-u)(1-\beta^2)^{1/2}, \tag{31}$$

where u is given by (27). Introducing cylindrical polar coordinates (r, θ, z) and the canonically conjugate momenta (p_r, p_θ, p_z), one notices that L is independent of $\dot{r}$ and of θ. The independence of $\dot{r}$ implies that one Euler–Lagrange equation is $\delta L/\delta r = 0$, yielding

$$2\beta^2 = u(1+\beta^2), \tag{32}$$

where one notes that $\delta u/\delta r = -(d/r_{12}^2)(\delta r_{12}/\delta r) = -u/r_{12}$ (since $r_{12}=r_1+r_2$) and $\delta\beta/\delta r = \beta/r$. Because L does not depend on θ, a second Euler–Lagrange equation yields the immediate integral

$$p_\theta = r\gamma\beta mc(1-u) = \text{constant}. \tag{33}$$

Whereas (33) is in agreement with $|\boldsymbol{r}\times\boldsymbol{p}| = \text{constant}$, where $\boldsymbol{p}$ is given by (29), the equation of motion (32) is in contradiction to (28). The reason, presumably, is the assumption of (18) rather than (10) when proceeding from (25) to (29). This assumption also is made in obtaining the results (19), (20) and (22).

One might argue that the Lagrangian (31) is but an approximation to

$$L = -mc^2 \exp(-u)(1-\beta^2)^{1/2}. \tag{34}$$

Then the equation of motion is indeed (28), but (33) becomes

$$r\gamma\beta mc \exp(-u) = \text{constant}. \tag{35}$$

A Lagrangian similar to (31) and (34) has been investigated by Fujigaki and Kojima (1978).

The Lagrangians (30), (31) or (34) are incomplete, however, without a field contribution. If one considers the total field of both particles, then the total Lagrangian should be $\int \mathscr{L}\, \mathrm{d}^3x$, where

$$\mathscr{L} = (8\pi)^{-1}(E^2 - B^2) - (\rho\varphi - \boldsymbol{j} \cdot \boldsymbol{A}). \tag{36}$$

Here $(\boldsymbol{j}, \rho)$ is the source 4-current density. Maxwell's equations follow from the variational principle $\delta \int \mathscr{L}\, \mathrm{d}^3x = 0$ treating the components of the 4-potential $(\boldsymbol{A}, \varphi) \equiv A^\alpha$ as independent variables (e.g. Goldstein 1959). $(\boldsymbol{r}, t) \equiv x^\alpha$ now behave as parameters which can be held constant for the variation-like λ when the space–time curve $x^\alpha(\lambda)$ is varied for the particle Lagrangian. The variation yields

$$\int_{t_1}^{t_2} \int \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta \mathscr{L}}{\delta(\delta A^\alpha/c\delta t)}\right) + \sum_{i=1}^{3} \frac{\mathrm{d}}{\mathrm{d}x^i}\left(\frac{\delta \mathscr{L}}{\delta(\delta A^\alpha/\delta x^i)}\right) - \frac{\delta \mathscr{L}}{\delta A^\alpha}\right] \delta A^\alpha \, \mathrm{d}^3x \, \mathrm{d}t = 0. \tag{37}$$

In order to form the Hamiltonian density $\mathscr{H}$, we require the variable canonically conjugate to $\delta \boldsymbol{A}/c\delta t$. This is

$$\mathscr{P} = \delta\mathscr{L}/\delta(\delta \boldsymbol{A}/c\delta t) = -\boldsymbol{E}/4\pi. \tag{38}$$

Then

$$\mathscr{H} = \mathscr{P} \cdot \delta \boldsymbol{A}/c\delta t - \mathscr{L} = (8\pi)^{-1}(E^2 + B^2 + 2\boldsymbol{E} \cdot \nabla\varphi) + (\rho\varphi - \boldsymbol{j} \cdot \boldsymbol{A}). \tag{39}$$

By use of $\nabla \cdot (\varphi \boldsymbol{E}) = \nabla\varphi \cdot \boldsymbol{E} + \varphi \nabla \cdot \boldsymbol{E}$ followed by Gauss's theorem and $\nabla \cdot \boldsymbol{E} = 4\pi\rho$, one finds that

$$H = \int \mathscr{H}\, \mathrm{d}^3x = (8\pi)^{-1} \int (E^2 + B^2)\, \mathrm{d}^3x - \int \boldsymbol{j} \cdot \boldsymbol{A}\, \mathrm{d}^3x. \tag{40}$$

Now we superimpose the fields of charges e_1 and e_2 in isolation. The terms $k \int (E_1^2 + B_1^2)\, \mathrm{d}^3x - \int \boldsymbol{j}_1 \cdot \boldsymbol{A}_1\, \mathrm{d}^3x$ provide the mechanical energy of e_1, namely $\gamma_1 mc^2$ (see Rohrlich 1960). A similar origin is found for the mechanical energy of e_2. For the interaction energy one obtains

$$V = 2k \int (\boldsymbol{E}_1 \cdot \boldsymbol{E}_2 + \boldsymbol{B}_1 \cdot \boldsymbol{B}_2)\, \mathrm{d}^3x - \int (\boldsymbol{j}_1 \cdot \boldsymbol{A}_2 + \boldsymbol{j}_2 \cdot \boldsymbol{A}_1)\, \mathrm{d}^3x = \int (\varphi_1\rho_2 - \boldsymbol{j}_1 \cdot \boldsymbol{A}_2)\, \mathrm{d}^3x, \tag{41}$$

where use is made of (19) and (20).

By substitution from (15) and use of $\boldsymbol{j}_1 = \boldsymbol{\beta}_1\rho_1$, with $\rho_1 = e_1\delta(\boldsymbol{r} - \boldsymbol{r}_1)$, one finds

$$V = \gamma_1(1 - \beta_1^2)e_1e_2/r_{12} = -(1-\beta^2)^{1/2}umc^2. \tag{42}$$

The total energy of the system is

$$W = 2\gamma mc^2 - (1-\beta^2)^{1/2}umc^2 = 2(1-\beta^2)^{1/2}mc^2, \tag{43}$$

where the equation of motion (28) has been used to eliminate u. This result agrees with that found from a Fokker action principle (Schild 1963, Andersen and von Baeyer 1971, Bruhns 1973). Moreover, Dorling (1970) obtained this result for Dirac's theory of the hydrogen atom. It is a consequence of the virial theorem.

5. Bohr stationary orbits

Accepting the equation of motion (26) or (28), namely

$$2\gamma^2\beta^2 = u, \tag{44}$$

it is proposed to select stationary orbits by Bohr-quantising the *canonical* angular momentum for which we adopt expression (33). Then

$$2r\gamma\beta mc(1-u) = n\hbar, \tag{45}$$

remembering that the motions of the two charges are not independent. Introducing the fine structure constant $\alpha = e^2/\hbar c = 1/137$, one may write (45) as

$$\gamma\beta(1-u) = nu/\alpha, \tag{46}$$

where one recalls that $2r = r_{12}$ and $u = d/r_{12}$.

Elimination of u between (44) and (46) yields a quadratic equation for $\gamma\beta$ with roots

$$\gamma\beta = -(n/2\alpha)[1 \pm (1+2\alpha^2/n^2)^{1/2}]. \tag{47}$$

On the other hand, elimination of $\gamma\beta$ between (44) and (46) yields a quadratic equation for u with roots

$$u = (n/\alpha)^2[1+\alpha^2/n^2 \pm (1+2\alpha^2/n^2)^{1/2}]. \tag{48}$$

To first order in α^2 one solution is

$$\gamma'\beta' = \alpha/2n, \qquad u' = \alpha^2/2n^2, \qquad W' = 2(1-\beta'^2)^{1/2}mc^2 = (2-\alpha^2/4n^2)mc^2 \tag{49}$$

and the other solution is

$$-\gamma''\beta'' = n/\alpha, \qquad u'' = 2n^2/\alpha^2, \qquad W'' = 2(1-\beta''^2)^{1/2}mc^2 = (2\alpha/n)mc^2. \tag{50}$$

In each case we have used formula (43) for the energies. The minus sign in front of $\gamma''\beta''$ merely means that the angular momentum in the field, the $-\gamma\beta u$ term of (46), is dominant; it can be eliminated by choosing the opposite sign for β or for n.

The solution (49) describes the 'atom-like' states of positronium already familiar to us. Since $\gamma\beta \ll 1$, orbital motion is nonrelativistic, and, since $u \ll 1$, orbital diameters are large compared with d. Energies differ from $2mc^2$ by order α^2mc^2.

The solution (50) describes new states, which will be termed 'particle-like'. Since $\gamma\beta \gg 1$, orbital motion is ultrarelativistic, and, since $u \gg 1$, orbital diameters are small compared with d; in fact, d is the geometric mean of the orbital diameters for corresponding (same n) atom-like and particle-like states. Energies differ from zero (not $2mc^2$) by order αmc^2.

In order to appreciate better the circumstances under which ultrarelativistic states arise, it is instructive to replace equations (44) and (46) by

$$2\beta^2 = (k_1 - k_2\beta^2)u, \qquad \gamma\beta(1-k_3u) = nu/\alpha, \tag{51}$$

where k_1, k_2 and k_3 are numerical constants which take the values unity when (51) is specialised to (44) and (46). Elimination of u leads to a quadratic equation for $\gamma\beta$ with roots

$$\gamma\beta = (n/K\alpha)[1 \pm (1 - k_1 K\alpha^2/n^2)^{1/2}], \tag{52}$$

where $K = k_1 - k_2 - 2k_3$. The corresponding values for u are obtained from

$$u = 2\gamma\beta/[k_1 + (k_1 - k_2)\gamma\beta]. \tag{53}$$

To first order in α^2 the solutions are

$$\gamma'\beta' = k_1\alpha/2n, \qquad u' = k_1\alpha^2/2n^2 \tag{54}$$

and

$$\gamma''\beta'' = 2n/K\alpha, \qquad u'' = \begin{cases} 2(k_1 - k_2)^{-1} & (k_1 \neq k_2) \\ 2n^2(k_1 k_3^2 \alpha^2)^{-1} & (k_1 = k_2). \end{cases} \tag{55}$$

The following comments may be made. The atom-like states (54) depend only on k_1, and not at all on k_2 or k_3. Thus magnetic interactions do not affect the atom-like states. Secondly, the orbital diameters of the particle-like states are of order $\alpha^2 d$ when $k_1 = k_2$ and of order d when $k_1 \neq k_2$. Thirdly, only if K vanishes do the particle-like states not arise; when $k_1 = k_2$ we have $K = -2k_3$, and hence k_3 must not vanish. In other words, the interaction angular momentum is crucial for the ultrarelativistic particle-like states.

The ultrarelativistic particle-like states involve interaction energy which greatly exceeds rest energy. In the co-rotating frame we have $e^2/r_{12} \gg mc^2$, implying $r_{12} \ll d$ or $u \gg 1$ (strong interactions). As mentioned in § 1, electrodynamics, either classical or quantum, is an untried theory in the realm of strong interactions.

Electron spin has not been introduced explicitly. In quantum theory, spin effects emerge automatically from Dirac's relativistic wave mechanics; one may add Pauli spin matrices as an empirical extra to Schrödinger's wave mechanics, but it is superfluous and wrong to add them to Dirac's theory. In the case of classical electrodynamics one again expects spin effects to be implicit in fully relativistic equations. For example, it is not fortuitous that spin–orbit interaction is implicit in the Sommerfeld treatment of the relativistic hydrogen atom, the energy states being in exact accord with Dirac's theory. For this reason classical theories of electron spin of the Thomas type (Thomas 1927, Bargmann *et al* 1959, Bacry 1962) are suspect, and so is the treatment of the electromagnetic two-body system with spin by Schild and Schlosser (1965, 1968). In all these theories a spin angular momentum tensor σ_{ij} and a spin moment tensor μ_{ij} $(= g\sigma_{ij})$ are introduced empirically. Corben (1961a) shows that the procedure leads to wrong results, and that it is necessary to choose $g = 0$. Spin effects are found to be implicit in a classical *Zitterbewegung* (Corben 1961b). Finally Corben (1962) shows that radiative reaction may be removed from the equations of motion by renormalisation of $\dot{\sigma}_{ij}$ and $\dot{m}$. This points the way to dispensing completely with the empirical σ_{ij} (Browne 1970). For surveys of classical theories of electron spin see Nyborg (1962a, b) and Jackson (1976).

Nothing has been said about vacuum polarisation effects which should dominate the scene when interaction is strong. Like spin, vacuum polarisation is a relativistic effect; the negative energy states arise automatically when Schrödinger's quantum mechanics is replaced by Dirac's. In classical electrodynamics the description of those effects which in quantum theory would be attributed to vacuum polarisation must also be implicit in relativistic equations. The field $\gamma(r)$ which multiplies the potentials (15) and field strengths (16) has all the characteristics of an electric permittivity and magnetic permeability (both equal) for the medium. It is of interest to recall how hole theory alters the contributions to the energy of a free electron, as analysed by Weisskopf (1939). (i) Firstly, the energy in the electrostatic field, which diverges as $1/r$ in electron theory, diverges only as $\ln(d/\alpha r)$ in hole theory. (ii) Secondly, energy in the magnetic and the solenoidal electric fields, which vanishes in electron theory, diverges as $1/r^2$ in hole theory. (iii) Thirdly, the energy in oscillatory motion of the electron due to the zero-point field fluctuations diverges as $1/r^2$ in both theories. In hole theory the contributions (ii) and (iii) cancel to a logarithmically divergent term, and so the total energy diverges logarithmically. But the difference of the energy for two states of the electron is finite and gives rise to the radiative corrections.

6. The 'positronium unit'

Positronium in one of the ultrarelativistic particle-like states (50) will be termed a 'positronium unit' or simply 'unit'. In this section the properties of such a unit are considered.

From (50) one finds $W'' = 7.46n^{-1}$ keV. Thus the spectroscopy of the ultrarelativistic states will be in the x-ray part of the spectrum. For example, 2-photon decay from the $n = 1$ state would yield quanta of energy 3.73 keV. A transition from the $n = 2$ to the $n = 1$ state yields a quantum of energy 3.37 eV, differing from 3.73 eV due to recoil. But radiative transitions will be exceedingly improbable because the orbital diameter r''_{12} is very small compared with the wavelength. Comparing the probability of 2-photon decay from the $n = 1$ ultrarelativistic state with that from the $n = 1$ nonrelativistic state, one expects a ratio $(k''/k')^3(r''_{12}/r'_{12})^2 = \alpha^{11}/16 = 1.96 \times 10^{-25}$, where k'' and k' are the wavevectors for the emitted photons given respectively by $\hbar ck' = 2mc^2$ and $\hbar ck'' = 2\alpha mc^2$; the ratio of orbital diameters is obtained from (49) and (50), namely $r''_{12}/r'_{12} = u'/u'' = \alpha^4/4$. The probability of the 2-photon decay from the familiar atom-like state is known to be $8 \times 10^9\ \mathrm{s}^{-1}$, and hence that from the particle-like state will be $1.6 \times 10^{-15}\ \mathrm{s}^{-1}$. For comparison, the probability of a transition between the hyperfine levels of the ground state of the H atom is $2.85 \times 10^{-15}\ \mathrm{s}^{-1}$; the transition is responsible for the chief spectral line at 21 cm in radio astronomy.

Although electrically neutral, the positronium unit in the singlet state should have a magnetic moment $\boldsymbol{\mu}''$, because the opposite spins of the electron and the positron imply parallel spin magnetic moments. No orbital moment arises, since the net orbital current vanishes. But it would be wrong to equate $\boldsymbol{\mu}''$ to two Bohr magnetons, because we deal with a situation in which interaction energy $-e^2/r''_{12}$ greatly exceeds rest energy mc^2. If one replaces the gyromagnetic ratio e/mc by e/m^*c, where $2m^*c^2 = 2mc^2 - e^2/r''_{12} \simeq -e^2/r''_{12}$, then the magnetic moment would be

$$\mu'' = |(e/m^*c)(n\hbar)| \simeq ner''_{12}/\alpha = \alpha ed/2n. \tag{56}$$

Here the angular momentum $n\hbar$ is associated with the unit.

Another argument substantiates this result. When an explicit spin magnetic moment $\frac{1}{2}\boldsymbol{\mu}''$ is assigned to charge 1, there will arise at the position of charge 2 a vector potential $\boldsymbol{A} = \frac{1}{2}\boldsymbol{\mu}'' \times \boldsymbol{r}''_{12}/r''^3_{12}$ in the co-rotating frame. Due to charge 1, charge 2 then has 'potential angular momentum' $\boldsymbol{r}''_{12} \times e\boldsymbol{A}/c = e\boldsymbol{\mu}''/2cr''_{12}$. Equating the latter to $\frac{1}{2}n\hbar$, one finds $\mu'' = ner''_{12}/\alpha$, in agreement with (56).

Angular momentum $n\hbar$ has been assigned to the positronium unit. But whether n is integral or half-integral remains to be decided. According to Beers (1972) the Wilson–Sommerfeld quantum numbers are integers only when the corresponding coordinate is ignorable, and otherwise equal an integer plus one-half. Now n is the sum of the Wilson–Sommerfeld quantum numbers, and if only r is non-ignorable it follows that n should be an integer plus one-half. The angular momentum of the positronium unit would then be $\frac{1}{2}\hbar$ in the state of lowest angular momentum. Regarding the half-integral values of n for the harmonic oscillator, see Boyer (1978).

Half-integral angular momentum of orbital origin implies a state of imaginary parity (Roman 1964). The possible intrinsic parities for fermions are ± 1 and $\pm \mathrm{i}$, the reason being that successive space inversions amount to operation by the identity matrix, and for Dirac spinors the 4×4 identity matrix has eigenvalues ± 1 (Roman 1964). But the parity of a fermion–antifermion pair always is -1.

If the spin of the positronium unit is half-integral, there arises the possibility of identifying it with the neutrino. Depending on the sense of orbital rotation relative to that of μ'' (not hitherto specified), the neutrino may be of electron or muon type. That is, we would assign intrinsic parities $+\mathrm{i}$ and $-\mathrm{i}$ to the ν_e and ν_μ, or vice versa. The parity of a ν_e, $\bar{\nu}_e$ pair, and also of a ν_μ, $\bar{\nu}_\mu$ pair, would be -1 as required, the antineutrino having the same parity as the neutrino. Whether to regard the positronium unit as a neutrino as it stands, or as a system comprising an electron, a positron and a neutrino, remains an open question, however.

The above model for the neutrino has the attractive consequence that a neutrino can interact electromagnetically due to its weak magnetic moment. If indeed all matter is built up solely from electrons and positrons, this would seem a necessary property of the neutrino. The model predicts a finite moment equal to α^2/n Bohr magnetons and a finite rest mass $2\alpha m/n$ for the neutrino. Experimental tests of these predictions may be devised, or more valuably of μ''/m'' which is independent of n. Recently experimental evidence for a finite neutrino mass has been reported tentatively, a value between 14 and 40 eV being mentioned, but the uncertainties associated with such a measurement are still too great. Previously, only upper limits for the neutrino mass had been established — 0.6 MeV for ν_μ and 60 eV for ν_e (Shrum and Ziock 1971). The predicted mass lies between 7.46 keV and 0 depending on the value of n.

The consequences of assigning imaginary intrinsic parity to the neutrino deserve more detailed investigation. Assuming that the Hamiltonian for a system of particles at positions $\boldsymbol{r}_k$ does not contain a term with the property $V(-\boldsymbol{r}_k) = \pm \mathrm{i} V(\boldsymbol{r}_k)$, the real or imaginary character of the parity (parity class) would be a conserved quantity. Then, from the decay products of any composite particle, the parity class of that particle can be inferred unambiguously. It turns out, after a survey of the parity classes of particles, that charged fermions and neutral bosons have real parity, while neutral fermions and charged bosons have imaginary parity. Various decay modes consistently yield the same parity class.

Of course identification of the positronium unit with the neutrino is a speculation. The ultrarelativistic states of positronium will exist whether or not one takes this step. Evidence in support of the identification comes from the argument in § 7.

7. The 'magnetium' system

A system in which one or more positronium units orbit about a charge under spin–orbit force will be termed 'magnetium'. The simplest system, dynamically, comprises two units orbiting at diametrically opposite positions (instantaneous positions in the centre-of-mass frame) around a stationary charge e located at the centre of mass 0. The electric field of the central charge, $E = \gamma e/r^2$ (since it rotates and is therefore given by (16)), interacts with the electric dipole moment $\boldsymbol{\epsilon}''$ due to motion with velocity $\boldsymbol{\beta}c$ of $\boldsymbol{\mu}''$ ($\boldsymbol{\epsilon}'' = \boldsymbol{\beta} \times \boldsymbol{\mu}''$). It is assumed that $\boldsymbol{\mu}''$ is normal to the orbital plane, and hence is a Lorentz invariant. Then $\boldsymbol{\epsilon}''$ is in the radial direction. The interaction energy V and the force F on a unit are given by

$$V = \boldsymbol{\epsilon}'' \cdot \boldsymbol{E} + \boldsymbol{\mu}'' \cdot \boldsymbol{B}, \qquad F = \epsilon'' \nabla_r E + \mu'' \nabla_r B, \tag{57}$$

where one notes that $\boldsymbol{B}$ does not vanish since the field rotates (see (16)). From (16) one has $\boldsymbol{B} = \boldsymbol{\beta} \times \boldsymbol{E}$, and since $\boldsymbol{\epsilon}'' = \boldsymbol{\beta} \times \boldsymbol{\mu}''$ it follows that $V = 0$. But the force does not vanish provided that $\beta(r)$ is treated as a field. Assuming that $\delta\beta/\delta r = \beta/r$, we find the results

$$V = 0, \qquad F = \gamma\beta\mu'' e/r^3. \tag{58}$$

The force between the units can be neglected, being of order μ''^2 which is small compared with $er\mu''$.

The equation for a circular orbit is

$$\gamma\beta^2 m'' c^2/r = \gamma\beta\mu'' e/r^3. \tag{59}$$

Since $m'' = 2\alpha m/n$ and $\mu'' = \alpha ed/2n$, this equation yields the simple result

$$\beta^{1/2} r = d/2. \tag{60}$$

Bohr quantisation of canonical angular momentum yields

$$2|\boldsymbol{r} \times (\gamma\boldsymbol{\beta} m'' c + e\boldsymbol{A}/c)| = \bar{n}\hbar, \tag{61}$$

where $\boldsymbol{A} = \gamma\boldsymbol{\mu}'' \times \boldsymbol{r}/r^3$. Hence

$$2\gamma\beta m'' cr + 2e\gamma\mu''/cr = 4\gamma\beta m'' cr = \bar{n}\hbar, \tag{62}$$

where use has been made of (59) in order to eliminate μ''. By substitution from (60) one finds, with $\beta = 1$,

$$\gamma = n\bar{n}/4\alpha^2. \tag{63}$$

For the energies of these ultrarelativistic states one has

$$W = 2\gamma m'' c^2 + mc^2 + V = (\bar{n}/\alpha + 1)mc^2. \tag{64}$$

Probably $\bar{n}' + \frac{1}{2}$ should replace $\bar{n}$, the new $\bar{n}'$ being integral, because we deal with a fermion; the spin angular momentum of the central charge should be included on the right-hand side of (61). Then for $\bar{n}' = 1$ the result (64) yields $W = 206.5mc^2$. This is remarkably close to the rest energy of the muon, $206.8mc^2$.

In regard to charge, spin, and rest energy the two-unit magnetium system seems to afford a satisfactory model for the muon. The decay of the muon also is explained. One of the positronium units in orbit around the central charge must be an electron neutrino and the other a muon neutrino; the reason being that the Pauli exclusion principle demands opposite spins, while attraction to the central charge demands parallel moments $\boldsymbol{\mu}''$. Hence the observed decay, $\mu^+ \to e^+, \bar{\nu}_\mu, \nu_e$, is to be regarded as a simple decomposition into constituents.

If only one unit orbits around the charge, the dynamics are somewhat complicated by the fact that the charge no longer is at rest at the centre of mass 0. If subscript 1 refers to the unit and subscript 2 to the charge e, then the vanishing relative to the centre-of-mass frame of the linear momentum of the system yields

$$\gamma_1 \boldsymbol{\beta}_1 m'' c + \gamma_2 \boldsymbol{\beta}_2 mc + e\gamma_1 \boldsymbol{\mu}'' \times \boldsymbol{r}_{12}/cr_{12}^3 = 0, \tag{65}$$

with $c\boldsymbol{\beta}_1 = \boldsymbol{\omega} \times \boldsymbol{r}_1$ and $c\boldsymbol{\beta}_2 = \boldsymbol{\omega} \times \boldsymbol{r}_2$. For attraction one requires that $(e\boldsymbol{r}_{21}/r_{21}^3) \cdot \boldsymbol{\beta}_1 \times \boldsymbol{\mu}'' < 0$, which implies that $\boldsymbol{\beta}_1 \cdot \boldsymbol{\mu}'' \times \boldsymbol{r}_{12} < 0$; hence the first and third terms of (65) have the same sign. The equation of motion expresses that the centrifugal force on either particle balances the interparticle force:

$$\gamma_1 \beta_1^2 m'' c^2/r_1 = \gamma_1 \beta_1 \mu'' e/r_{12}^2 r_1 = \gamma_2 \beta_2^2 mc^2/r_2. \tag{66}$$

Since motion of both particles is ultrarelativistic, we have $\beta_1 \simeq \beta_2 \simeq 1$. Hence $r_1 \simeq r_2$, so that the centre of mass is at the midpoint to a good approximation (despite the unequal rest masses). The equality of the centrifugal forces in (66) then implies that $\gamma_1 m'' \simeq \gamma_2 m$. The interparticle force in (66) is again obtained from (57), the gradient being with respect to r_1. Substituting for m'' and for μ'', one now finds

$$\beta^{1/2} r_1 = d/4. \tag{67}$$

Bohr quantisation of the canonical angular momentum now gives

$$|\boldsymbol{r}_1 \times \gamma_1 \boldsymbol{\beta}_1 m'' c + \boldsymbol{r}_2 \times \gamma_2 \boldsymbol{\beta}_2 mc + \boldsymbol{r}_{21} \times (e\gamma_1 \boldsymbol{\mu}'' \times \boldsymbol{r}_{12}/cr_{12}^3)| = \bar{n}\hbar. \tag{68}$$

The first two terms give equal contributions. The third term, after use of (65), gives a contribution equal to the sum of the other contributions. Hence (68) reduces to

$$4\gamma_1 \beta_1 m'' c r_1 = \bar{n}\hbar, \tag{69}$$

which yields, with the help of (67) and $\beta_1 = 1$,

$$\gamma_1 = n\bar{n}/2\alpha^2. \tag{70}$$

For the energies one finds

$$W = \gamma_1 m'' c^2 + \gamma_2 mc^2 + V = 2\gamma_1 m'' c^2 = (2\bar{n}/\alpha) mc^2. \tag{71}$$

The state $\bar{n} = 1$ has energy $274mc^2$, which is close to the rest energy of the charged pion, $273.1mc^2$.

In regard to charge, spin, and rest energy the one-unit magnetium system affords a satisfactory model for the charged pions. The observed decay, $\pi^+ \to \mu^+, \nu_\mu$, can be explained by the creation of a $\nu_\mu, \bar{\nu}_\mu$ pair with retention of $\bar{\nu}_\mu$ and release of ν_μ; the decay particle then consists of $e^+, \nu_e, \bar{\nu}_\mu$, which is just the muon system according to our previous model.

The π^0 presumably is a combination of two one-unit magnetium systems, one with e^- and the other with e^+. One envisages a molecular-type complex; since the energy of the two non-interacting systems would be $(4\bar{n}/\alpha)mc^2$, the binding energy must be somewhat greater than $(2\bar{n}/\alpha)mc^2$.

There remains the question of why the muon should be very much less reactive than the charged pion. The reason may be a closed shell in the case of the two-unit magnetium. Then the relationship between the muon and the charged pion becomes analogous to the relationship between an inert gas atom and a monovalent metal atom; the former is unreactive in comparison with the latter. The significance of closed shells of course is well known also in nuclear physics, the literature being traceable from a communication by Pauling (1965).

8. 'Trionium'

A system in which two charges of one sign orbit around a central stationary charge of the opposite sign will be termed 'trionium'. One envisages a common circular orbit for the moving charges which have diametrically opposed positions instantaneously for the centre-of-mass frame. In general the moving charges might follow confocal ellipses.

Assuming co-rotating fields as given by (15) and (16), the force which maintains each charge in orbit is

$$F=(1-\beta^2)^{1/2}(e^2/r^2-e^2/r_{12}^2)=(1-\beta^2)^{1/2}3e^2/r_{12}^2. \tag{72}$$

The equation of motion, in terms of $u=d/r_{12}$, is

$$2\gamma^2\beta^2=3u. \tag{73}$$

Bohr quantisation of canonical angular momentum yields

$$2|\boldsymbol{r}\times(\gamma\boldsymbol{\beta}mc+e\mathbf{A}/c)|=\bar{n}\hbar, \tag{74}$$

where $\boldsymbol{A}$ is obtained by Lorentz transformation from the co-rotating frame in accordance with (11); that is,

$$\boldsymbol{A}=\gamma\boldsymbol{\beta}\varphi', \qquad \varphi'=-e/r+e/r_{12}=-e/2r. \tag{75}$$

Hence (74) becomes

$$\gamma\beta(1-u)=\bar{n}/\alpha. \tag{76}$$

Now one solves (73) and (76) for $\gamma\beta$ and u in the usual manner. Since these equations can be treated as a special case of (51) with $k_1=k_2=3$ and $k_3=1$, the two solutions are given by (54) and (55), namely

$$\gamma'\beta'=3\alpha/2n, \qquad u'=3\alpha^2/2n^2 \tag{77}$$

and

$$\gamma''\beta''=n/\alpha, \qquad u''=2n^2/3\alpha^2. \tag{78}$$

The energies of the states are given by

$$W=2\gamma mc^2+mc^2+V, \tag{79}$$

where

$$V=(-2e^2/r+e^2/r_{12})\gamma(1-\beta^2). \tag{80}$$

From (72) one notes that $V=2rF$, and hence $V=-2\gamma\beta^2mc^2$ from the equation of motion. By substitution into (79) one finds again

$$W=2(1-\beta^2)^{1/2}mc^2+mc^2. \tag{81}$$

In the case of the ultrarelativistic states $W\simeq mc^2$. Thus the ultrarelativistic states differ inappreciably from the rest energy of the central charge.

The circumstances under which the result (81) obtains have been examined by Andersen and von Baeyer (1971), and see also Dorling (1970). So long as we have $V=2rF$ with $F=\gamma\beta^2mc^2/r$, it is clear that $2\gamma mc^2+V=2(1-\beta^2)^{1/2}mc^2$.

In the case of trionium the orbiting charges provide a current $2e(\omega/2\pi)=e\beta c/\pi r$. The magnetic field due to this current evaluated at the centre of the mass is $B=2e\beta/r^2$, bearing in mind that $\gamma(0)=1$. If one attributes to the central charge an explicit spin magnetic moment μ_0, then the coupling energy would be $\mu_0 B = 2e\mu_0\beta/r^2$. Essentially this is the type of interaction responsible for hyperfine splitting of spectra in the atomic domain, for which a classical treatment has been given by Ferrell (1960). But as discussed at the end of § 5, spin effects should be *implicit* in fully relativisitic equations, whether classical or quantum. Consequently one might expect to find $\mu_0 B = 2\gamma e^2/r$, which yields $\mu_0 = \gamma er = ner/\alpha$. The same result is obtained if one replaces the gyromagnetic ratio e/mc by e/m^*c, where $m^*c^2 = mc^2 - e^2/r$ (for the co-rotating frame); this was the procedure leading to (56). Now one finds $\mu_0 = (e/m^*c)(n\hbar) \simeq -ner/\alpha$.

9. 'The stack'

The term 'stack' refers to a system in which pairs of identical charges orbit around a central charge. The radii of the orbits increase because the orbital magnetic moment for inner orbits increases, giving adequate restraining force for looser orbits; the charges alternate in sign for successive orbits in the sequence, so that the net charge is always $\pm e$. Another way to describe the system is as follows. Consider trionium, with say central charge $+e$ and orbital charges $-e$ giving net charge $-e$. Then introduce a pair of charges $+e$ in orbit around the entire system and subject to the magnetic force $e\boldsymbol{\beta} \times \boldsymbol{B}_1$, where $\boldsymbol{B}_1$ is the magnetic field due to orbital motion of the trionium electrons; this force dominates. The additional pair of charges produces a larger orbital magnetic moment which can be used to couple a further pair of charges in an even larger orbit, and so on. Of course all orbits are assumed to be co-planar.

Let the radii of successive orbits be $r_1, r_2, \ldots$, and let the magnetic moments associated with the orbits be $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \ldots$. In the case of orbit 1, that of trionium, there is no explicit central moment; the spin moment of the central charge (denoted $\boldsymbol{\mu}_0$ above) is implicit. In the case of orbit 2 the central moment is that of orbit 1, namely $\boldsymbol{\mu}_1$. Hence the equation of motion for the charges in orbit 2 is

$$\gamma\beta^2 mc^2/r_2 = |e\boldsymbol{\beta} \times \gamma\boldsymbol{\mu}_1/r_2^3| + \gamma(1-\beta^2)3e^2/4r_2^2. \tag{82}$$

The second term on the right-hand side of (82) can be neglected for ultrarelativistic motion, since $\beta \simeq 1$, and the first term can be simplified by using

$$\mu_1 = (\pi r_1^2)(2e\omega_1/2\pi) = e\beta_1 r_1 \simeq er_1. \tag{83}$$

Then (82) reduces to

$$r_2^2 = r_1 d. \tag{84}$$

Bohr quantisation of canonical angular momentum gives

$$2|\boldsymbol{r}_2 \times (\gamma\boldsymbol{\beta}mc + e\boldsymbol{A}/c)| = n\hbar, \qquad \boldsymbol{A} = -\gamma\boldsymbol{\beta}e/2r_2 + \gamma\boldsymbol{\mu}_1 \times \boldsymbol{r}_2/r_2^3. \tag{85}$$

Since the equation of motion yields $e\mu_1/r_2^2 c \simeq \gamma\beta mc$, one may simplify (85) to

$$|2r_2\gamma\beta(2 - d/2r_2)| = nd/\alpha. \tag{86}$$

When $r_2 \ll d/4$, the quantisation condition (86) yields $\gamma = n/\alpha$ irrespective of the value of r_2.

The interaction energy essentially is that for trionium, with an additional term to represent the magnetic coupling between the orbital currents. One has

$$V = \gamma(1-\beta^2)(-2e^2/r_2 + e^2/2r_2) - \mu_1(2\beta e/r_2^2) + \epsilon_2. \tag{87}$$

The first term is essentially (80). In the second $2\beta e/r_2^2$ is the field due to the outer orbital current, and ϵ_2 corrects for the non-dipole character of the inner orbital moment. Because of the virial theorem (which leads to formula (81) for energy) we shall see that ϵ_2 is important. With the help of the equation of motion (82) one proceeds to simplify expression (87) for V in the usual manner; one finds

$$V = -2r_2F = -2\gamma\beta^2mc^2, \tag{88}$$

where F is the restraining force. Hence the total energy is given by

$$W = 2\gamma mc^2 + mc^2 + V = 2(1-\beta^2)^{1/2}mc^2 + mc^2 + \epsilon_2. \tag{89}$$

It remains to estimate ϵ_2. The energy in the magnetic field due to two concentric circular currents J_1 and J_2 is

$$(L_{11}J_1^2 + 2L_{12}J_1J_2 + L_{22}J_2^2)/2c, \tag{90}$$

where L_{rs} are coefficients of self and mutual inductance. For the interaction energy one has

$$\Delta V = L_{12}J_1J_2/c, \qquad L_{12} = \gamma c^{-1}\oint\oint \mathrm{d}\boldsymbol{l}_1\cdot\mathrm{d}\boldsymbol{l}_2/s_{12}, \tag{91}$$

where the permeability of the medium is taken to be γ, and s_{12} is the distance between the circuit elements $\mathrm{d}\boldsymbol{l}_1$ and $\mathrm{d}\boldsymbol{l}_2$. This well-known calculation yields

$$L_{12} = 4\pi c^{-1}\gamma\lambda(r_1r_2)^{1/2}\int_0^{\pi/2}\frac{2\sin^2\varphi - 1}{(1-\lambda^2\sin^2\varphi)^{1/2}}\,\mathrm{d}\varphi, \tag{92}$$

where $\lambda^2 = 4r_1r_2/(r_1+r_2)^2$. The integral involves elliptic functions, but in the present case we have $r_2 \gg r_1$, so that $\lambda^2 \simeq 4r_1/r_2 \ll 1$ and hence we can approximate to the integral by expanding in powers of λ^2. This procedure leads to

$$L_{12} = 2\pi^2c^{-1}\gamma(r_1^2/r_2)(1+3r_1/r_2+\ldots). \tag{93}$$

Since $J_1 = 2e(\omega_1/2\pi)$ and $J_2 = -2e(\omega_2/2\pi)$, one finds

$$\Delta V = 2\gamma e^2(r_1/r_2^2)(1+3r_1/r_2). \tag{94}$$

Hence

$$\epsilon_2 = 6\gamma e^2r_1^2/r_2^3 = 6(r_2/d)nmc^2/\alpha, \tag{95}$$

where use has been made of $r_2^2 = r_1d$ and also $\gamma = n/\alpha$, a consequence of (86). Clearly the energy due to magnetic interaction of the orbital currents cannot be neglected.

If one assumes that r_1 is the outermost trionium orbit with radius $3\alpha^2d/4$, then $r_2/d = 3^{1/2}\alpha/2$ and $\epsilon_2 = 5.2nmc^2$. Proceeding outwards, successive radii in accordance with the recurrence relation $(r_i/d)^2 = r_{i-1}/d$ work out to be as follows: $r_i/d = 1/25\,000$, $1/158$, $1/12.7$, $1/3.5$, $1/1.9$, $1/1.4$, $1/1.2$, $1/1.1$, etc. Thus for $i=5$ one has $r_5 \simeq d/2$. Since $\lambda^2 = 4r_{n-1}/r_n = 4r_n/d$, it is valid to assume $\lambda^2 \ll 1$ only out to $i=3$. The energy of the system arising from magnetic coupling between currents is $\epsilon_2+\epsilon_3+\epsilon_4+$ etc, but only out to about ϵ_3 can one rely on the approximation which yields (95). Clearly, however, the coupling can produce energy comparable with that of the proton.

Had we neglected the magnetic interaction energy one might simply have filled up the orbits of trionium, assigning a pair of identical charges to each orbit over the range of quantum numbers $n=1$ to N. One notes from (78) that the orbital radii decrease with n according to n^{-2}. The energy contributed by a pair of charges in orbit n is simply

$(\alpha/n)mc^2$ according to (78) and (81). Thus the total energy of the stack would be $(\alpha \ln N+1)mc^2$. In principle N could tend to infinity, the orbital radii tending to zero. This would yield a logarithmic divergence, which would demand some sort of cut-off based on cosmological constants (see e.g. Browne 1962, 1976, 1977b).

Clearly the stack has exceptional properties. One of these is likely to be a very high degree of stability; because the two charges in each orbit are identical (rather than particle and antiparticle), an internal annihilation of an electron and positron is unlikely. One wonders, therefore, if the stack might be a possible model for the proton. A considerably more detailed calculation will be required to evaluate the energy of the system.

10. Concluding remarks

The prospect of an electron–positron structure for all matter seems closer. Then only electromagnetic forces need be considered in particle physics. In this regard the very fact that many particles have energies close to a multiple of $137mc^2$ is significant, and the recently discovered upsilon meson, with spin 1, has energy very close to 137^2mc^2 ($=9.60$ GeV). In applying classical electrodynamics to situations where the interaction energy between charges greatly exceeds their rest energy, implying ultrarelativistic motion, one opens up a field of research which will require more thorough exploration. I should be surprised if the above development were free from error, but one cannot expect more when breaking completely new ground.

References

Andersen C M and von Baeyer H C 1971 *Am. J. Phys.* **39** 914–9
—— 1972 *Phys. Rev.* D **5** 802–13
Bacry H 1962 *Nuovo Cimento* **26** 1164–71
Bargmann V, Michel L and Telegdi V L 1959 *Phys. Rev. Lett.* **2** 435–6
Beers V 1972 *Am. J. Phys.* **40** 1139–46
Boyer T H 1975 *Phys. Rev.* D **11** 809–30
—— 1978 *Phys. Rev.* A **18** 1238–45
Breitenberger E 1968 *Am. J. Phys.* **36** 505–15
Browne P F 1962 *Nature* **193** 1019–21
—— 1966 *Nature* **211** 810–3
—— 1969 *Phys. Lett.* **29A** 588–9
—— 1970 *Ann. Phys., NY* **59** 254–8
—— 1976 *Int. J. Theor. Phys.* **15** 73–9
—— 1977a *J. Phys. A: Math. Gen.* **10** 727–44
—— 1977b *Found. Phys.* **7** 165–83
Bruhns B 1973 *Phys. Rev.* D **8** 2370–9
Calkin M G 1966 *Am. J. Phys.* **34** 921–5
—— 1971 *Am. J. Phys.* **39** 513–6
Corben H C 1961a *Phys. Rev.* **121** 1833–9
—— 1961b *Nuovo Cimento* **20** 1841–53
—— 1962 *Proc. Natl Acad. Sci. USA* **48** 387–95
Corum J F 1977 *J. Math. Phys.* **18** 770–6
—— 1980 *J. Math. Phys.* **21** 2360–2
Currie D G 1966 *Phys. Rev.* **142** 817–42
Dettman J W and Schild A 1954 *Phys. Rev.* **95** 1057–60
Dirac P A M 1950 *Can. J. Phys.* **2** 129–48

Dorling J 1970 *Am. J. Phys.* **38** 510–2
Driver R D 1963 *Ann. Phys., NY* **21** 122–42
Erlichson H 1970 *Am. J. Phys.* **38** 162–73
Fahnline D E 1977 *J. Math. Phys.* **18** 1006–11
Ferrell R A 1960 *Am. J. Phys.* **28** 484–6
Feynman R P 1948 *Rev. Mod. Phys.* **20** 367–87
Foldy L L and Wouthuysen S A 1950 *Phys. Rev.* **78** 29–36
Fujigaki M and Kojima S 1978 *Prog. Theor. Phys.* **59** 1330–45
Furry W H 1969 *Am. J. Phys.* **37** 621–36
Goldstein H 1959 *Classical Mechanics* (Reading, Mass.: Addison-Wesley) ch 11
Graham G M and Lahoz D G 1980 *Nature* **285** 154–5
Heisenberg W 1976 *Phys. Today* **29** 32–9
Hill R N 1967 *J. Math. Phys.* **8** 201–20
Jackson J D 1976 *Rev. Mod. Phys.* **48** 417–33
Konopinski E J 1978 *Am. J. Phys.* **46** 499–502
Kracklauer A F 1978 *J. Math. Phys.* **19** 808–11
Leutwyler H 1965 *Nuovo Cimento* **37** 556–67
Marshall T W 1963 *Proc. R. Soc.* A **276** 475–91
—— 1965 *Proc. Camb. Phil. Soc.* **61** 537–46
Milne E A 1948 *Kinematic Relativity* (Oxford: Clarendon) pp 216–9
Motz L 1962 *Phys. Rev.* **126** 378–82
Muller B, Rafelski J and Greiner W 1972a *Z. Phys.* **257** 62–77
—— 1972b *Z. Phys.* **257** 183–211
Nyborg P 1962a *Nuovo Cimento* **23** 47–62
—— 1962b *Nuovo Cimento* **26** 821–30
Page L 1929 *Phys. Rev.* **33** 572–83
Page L and Adams L I 1940 *Electrodynamics* (New York: Van Nostrand) (also 1965 (New York: Dover)) ch 3
Pauling L 1965 *Nature* **208** 174
de la Pena L and Cetto A M 1979 *J. Math. Phys.* **20** 469–83
de la Pena-Auerbach L and Garcia-Colin L S 1968 *J. Math. Phys.* **9** 916–21
Power E A 1966 *Am. J. Phys.* **34** 516–8
Rohrlich F 1960 *Am. J. Phys.* **28** 639–43
—— 1979 *Ann. Phys., NY* **117** 292–322
Roman P 1964 *Theory of Elementary Particles* 2nd edn (Amsterdam: North-Holland) pp 258–60
Schiff L I 1939 *Proc. Natl Acad. Sci. USA* **25** 391–5
Schild A 1963 *Phys. Rev.* **131** 2762–6
—— 1975 *Ann. Phys., NY* **93** 88–115
—— 1976 *Ann. Phys., NY* **99** 434–58
Schild A and Schlosser J A 1965 *J. Math. Phys.* **6** 1299–306
—— 1968 *J. Math. Phys.* **9** 913–5
Shrum E V and Ziock K O H 1971 *Phys. Lett.* **37B** 115–6
Smith P F 1965 *Nuovo Cimento* **38** 1824–34
Staruszkiewicz A 1968 *Acta Phys. Polon.* **33** 1007–11
—— 1970 *Ann. Phys., Lpz* **25** 362–7
Stephas P 1978 *Am. J. Phys.* **40** 360–5
Sternglass E J 1961 *Phys. Rev.* **123** 391–8
—— 1965 *Nuovo Cimento* **35** 227–60
Synge J L 1972 in *Magic without Magic* ed. J R Klauder (San Francisco: Freeman) pp 117–33
Thomas L H 1927 *Phil. Mag.* **3** 1–22
Van Dam H and Wigner E P 1965 *Phys. Rev.* **138** B1576–82
—— 1966 *Phys. Rev.* **142** 838–43
Weisskopf V F 1939 *Phys. Rev.* **56** 72–85
Welton T A 1948 *Phys. Rev.* **74** 1157–67
Wheeler J A and Feynman R P 1945 *Rev. Mod. Phys.* **17** 157–81